Optimization over Task Periods with Exact Schedulability Constraints

upon the papers
E. Bini, M. Di Natale, "Optimal task rate selection in fixed priority systems", RTSS'05, Miami
E. Bini, M. Di Natale, G. Buttazzo, "Sensitivity analysis for fixed-priority real-time systems", Real-Time Systems 2008

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The problem

- In control systems (and elsewhere), it is necessary to set the period of tasks, such that
  1. schedulability is preserved, and
  2. a given performance function is maximised (equivalently a given cost is minimized)
- Given n tasks $\mathcal{T} = \{\tau_1, \ldots, \tau_n\}$
  - Each task τ_i is modelled by its execution time C_i only
  - Implicit deadline $D_i = T_i$
- Find the task periods T_i such that

$$\begin{aligned} \text{maximize}_{T_1,\ldots,T_n} \quad & J(T_1, \ldots, T_n) \\ \text{such that} \quad & \mathcal{T} \text{ is schedulable} \end{aligned}$$

- Often convenient to formulate with task rates $f_i = \frac{1}{T_i}$
- Using vector notation: $\mathbf{C} = (C_1, C_2, \ldots, C_n)$, $\mathbf{T} = (T_1, T_2, \ldots, T_n)$, $\mathbf{f} = (f_1, f_2, \ldots, f_n)$

$$\begin{aligned} \text{maximize}_{\mathbf{f}} \quad & J(\mathbf{f}) \\ \text{such that} \quad & \mathbf{f} \in \mathcal{F} \end{aligned}$$

- $\mathcal{F} = \{\mathbf{f} : \mathcal{T} \text{ is schedulable}\}$

EDF & Fixed Priorities

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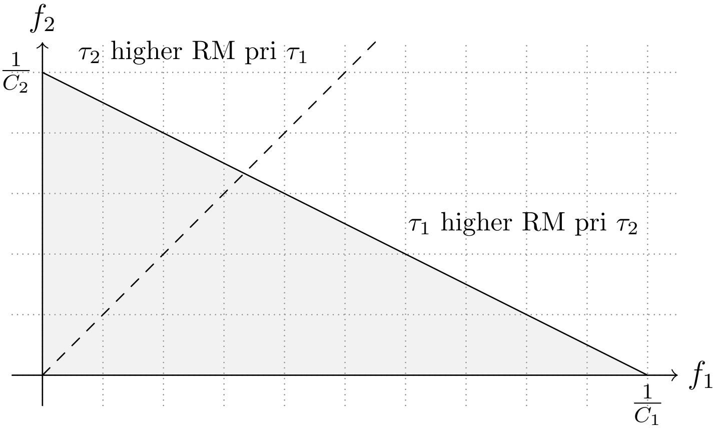
- Fixed Priority or Rate Monotonic?
  - RM: the rates $\mathbf{f}$ influences priorities



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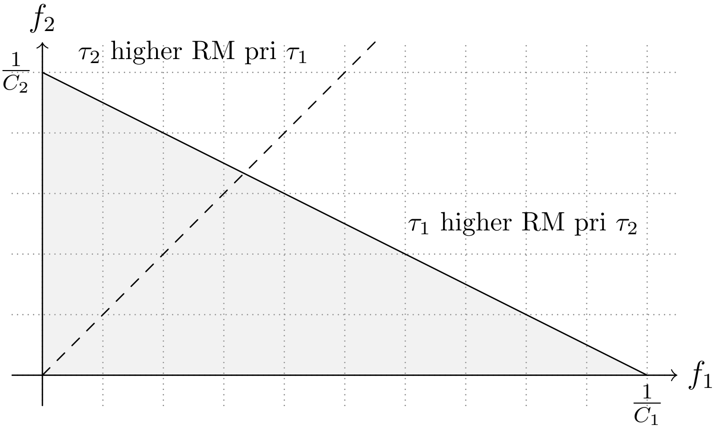


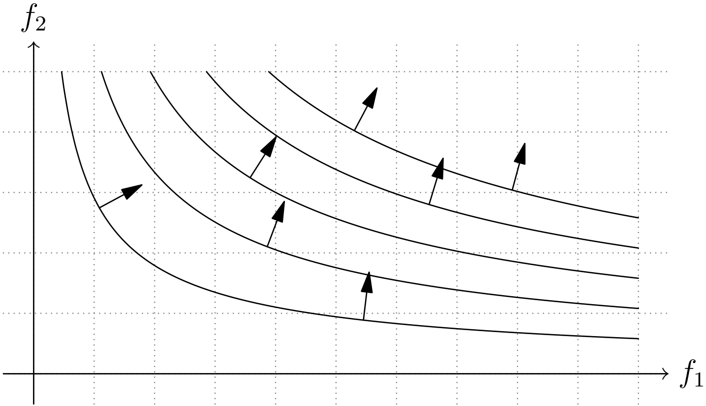
From now on: $\mathbf{f} \in \mathcal{F} \quad \Leftrightarrow \quad \mathcal{T}$ is RM-schedulable

The performance to be maximized

- The "performance" normally encodes:
  - control metrics (LQR, etc.)
  - Quality of Service
- "The faster the better"

$$\frac{\partial J}{\partial f_i} > 0$$

with $J(\mathbf{f})$, the performance



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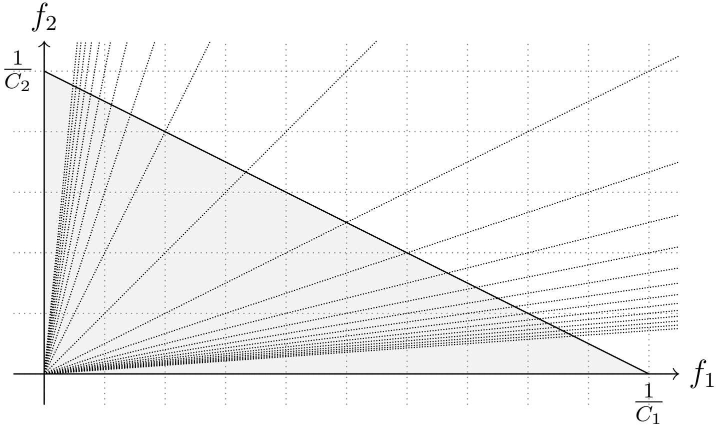

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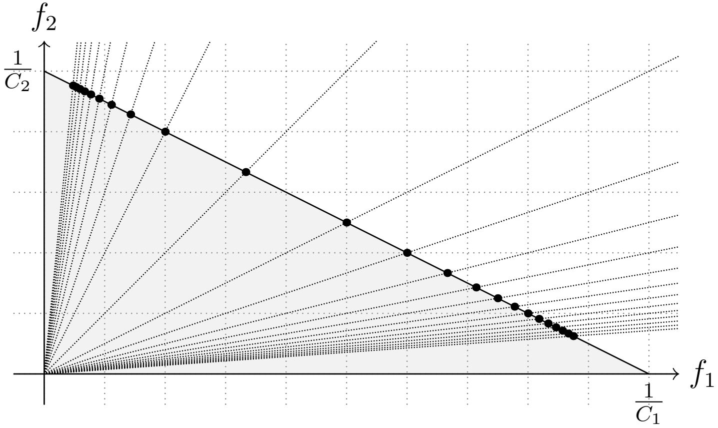

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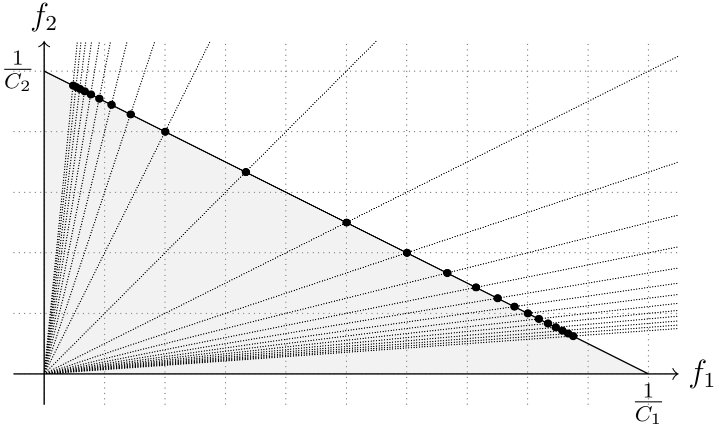

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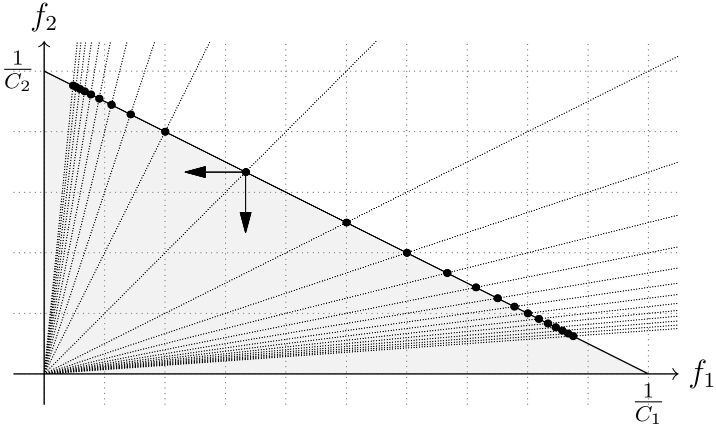

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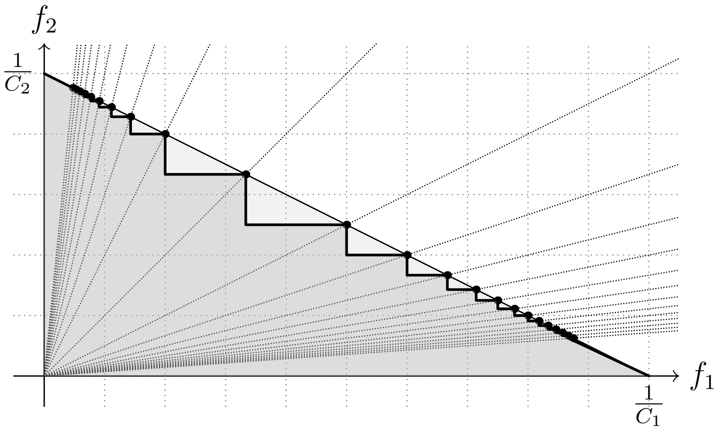
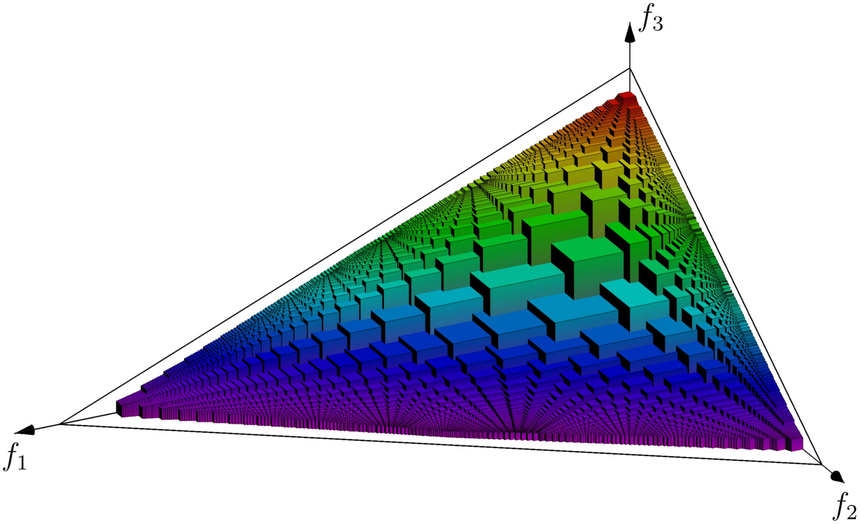


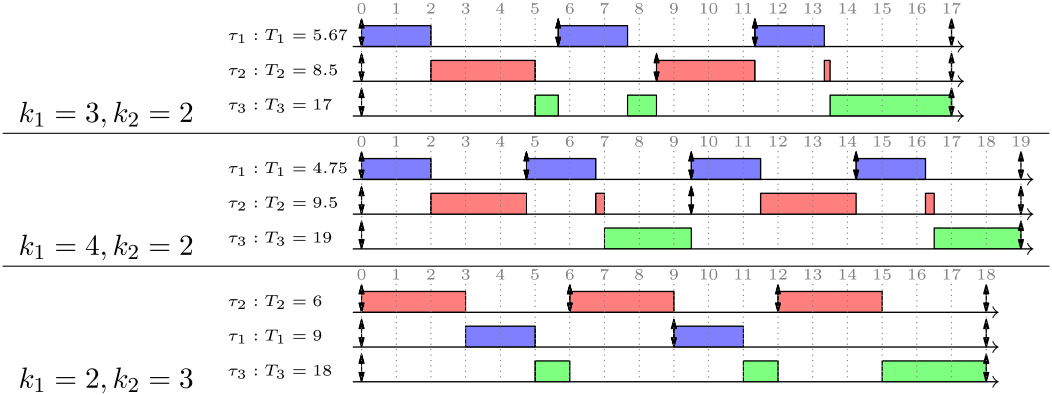
RM-feasible frequencies ($n = 3$)



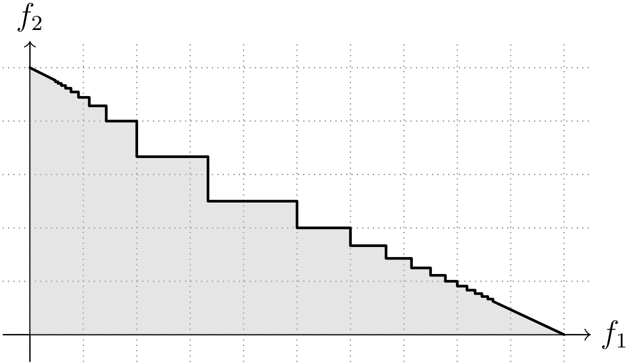
Understanding the solution at the vertices

- Vertices correspond to
  1. full utilization
  2. harmonic frequencies $T_n = \mathrm{lcm}\{T_i\}$ implies that
  3. response time R_n is such that $\frac{R_n}{T_i} = k_i$ for some $k_i \in \mathbb{N}$

Below, some schedules of harmonic+full-util cases, with $C_1 = 2$, $C_2 = 3$, and $C_3 = 5$



The optimization algorithm



- All vertices are local optima. Why?

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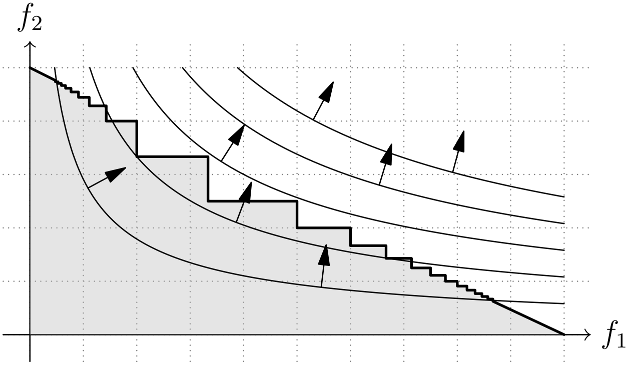

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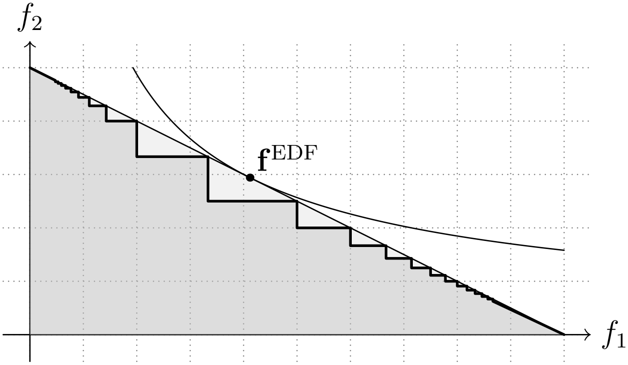

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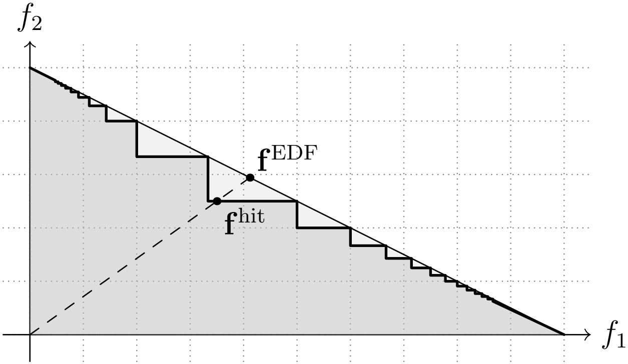

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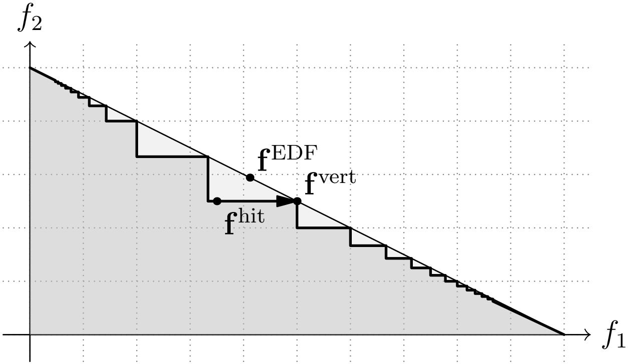

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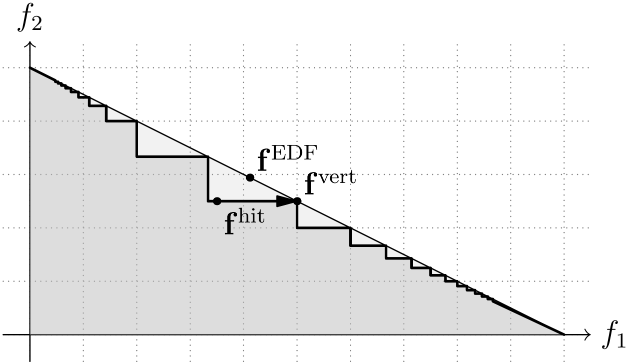

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4. Optimum ($\mathbf{f}^{\mathrm{RM}}$) found by branch and bound (if needed)

Final remarks

- Comparing the performance: $J(\mathbf{f}^{\mathrm{EDF}}) \geq J(\mathbf{f}^{\mathrm{RM}}) \geq J(\mathbf{f}^{\mathrm{vert}})$
- Experiments [RTSS'05] show that in average

$$J(\mathbf{f}^{\mathrm{RM}}) \approx 0.994\, J(\mathbf{f}^{\mathrm{EDF}}), \qquad J(\mathbf{f}^{\mathrm{vert}}) \approx 0.997\, J(\mathbf{f}^{\mathrm{RM}})$$

- Finding $\mathbf{f}^{\mathrm{vert}}$ does not require expensive branch and bound
- Bonus: the vertex solution $\mathbf{f}^{\mathrm{vert}}$ also has
  - hyperperiod equal to the maximum period
  - all tasks have very limited output jitter (good for control)

Bini, Di Natale, RTSS'05

- Space of RM-schedulable rates
- The optimization procedure

Bini, Di Natale, Buttazzo, RTSJ'08

- The "scaling down procedure" $(\mathbf{f}^{\mathrm{EDF}} \to \mathbf{f}^{\mathrm{hit}})$
- Using the response time to find $\mathbf{f}^{\mathrm{vert}}$